Question 1 **(5 marks)**

Determine the equation of the tangent to the curve defined $x^2 - xy + y^3 = 5$ at the point $(2, -1)$.

$$\frac{d(x^2 - xy + y^3)}{dx} = \frac{d(5)}{dx}$$

$$2x - \left(1 \times y + x\frac{dy}{dx}\right) + 3y^2\frac{dy}{dx} = 0$$ ✓ implicit differentiation

$$2x - y - x\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0$$

$$(3y^2 - x)\frac{dy}{dx} = y - 2x$$

$$\frac{dy}{dx} = \frac{y - 2x}{3y^2 - x}$$

✓ determines $\frac{dy}{dx}$

$$\therefore m = \frac{-1 - 2(2)}{3(1) - 2} = -5$$ ✓ determines gradient

$$\therefore -1 = -5(2) + c$$ ✓ determines constant term

$$\therefore c = 9$$

$$\therefore y = -5x + 9$$ ✓

Question 2 **(6 marks)**

A salad, which is initially at a temperature of 25°C, is placed in a refrigerator that has a constant temperature of 3°C. The cooling rate of the salad is proportional to the difference between the temperature of the refrigerator and the temperature, T, of the salad. That is, T satisfies the equation

$$\frac{dT}{dt} = -k(T-3)$$

(a) Show that $T = 3 + Ae^{-kt}$ satisfies this equation. (2 marks)

LHS: $\frac{dT}{dt} = -kAe^{-kt}$ ✓

RHS: $-k(3 + Ae^{-kt} - 3) = -kAe^{-kt}$ ✓

Since LHS = RHS, $T = 3 + Ae^{-kt}$ satisfies this equation.

(b) The temperature of the salad is 11°C after 10 minutes.
Find the temperature of the salad after 15 minutes. (4 marks)

$t=0$: $25 = 3 + A$

$A = 22$ ✓ determines A

$t=10$: $11 = 3 + 22e^{-10k}$

$\frac{8}{22} = e^{-10k}$ ✓ solve for e^{-10k} or k

$t=15$: $T = 3 + 22e^{-15k}$ ✓ correct connection

$= 3 + 22(e^{-10k})^{1.5}$

$= 3 + 22\left(\frac{4}{11}\right)^{1.5}$

$= 3 + 22\,\frac{4}{11}\sqrt{\frac{4}{11}}$

$= 3 + \frac{16}{\sqrt{11}}$ ✓

Question 3 **(4 marks)**

A particle is moving in simple harmonic motion in a straight line. Its maximum speed is 3 m/s and its maximum acceleration is 6 m/s^2.

Find the amplitude and the period of the motion.

$x = A\sin(kt)$

$\Downarrow$ $v = kA\cos(kt)$

$\Downarrow$ $a = -k^2 A\sin(kt)$

$\therefore$ $v_{max} = kA$ ✓ either

$a_{max} = k^2 A$

$\therefore$ $kA = 3$ ✓ either

$k^2 A = 6$

$\therefore$ $k = 2$

$A = 1.5$

$\therefore$ period $= \pi$ seconds ✓

Amplitude $= 1.5$ metres ✓

End of questions

Question 4 **(6 marks)**

Consider the differential equation

$$\frac{dy}{dx} = \frac{-x}{2y}$$

with initial values $x = 2$ and $y = 3$.

(a) Use Euler's method with a step size of 0.2 in the values of x to determine an approximate value of y when $x = 2.6$. (3 marks)

1st iteration: $y \approx 3 + \frac{-2}{6}(0.2) = 2.9\overline{3}$ ✓

2nd iteration: $y \approx 2.9\overline{3} + \frac{-2.2}{2(2.9\overline{3})} = 2.858\overline{3}$ ✓

3rd iteration: $y \approx 2.858\overline{3} + \frac{-2.4}{2(2.858\overline{3})} \approx 2.77$ ✓

(b) Determine the exact value of y when $x = 2.6$. (3 marks)

$$\frac{dy}{dx} = \frac{-x}{2y}$$

$$\int 2y\,dy = \int -x\,dx$$ ✓ separate varibles

$$y^2 = -\frac{1}{2}x^2 + c$$

$(x,y)=(2,3)$: $9 = -\frac{1}{2}(4) + c \quad \therefore \quad c = 11$ ✓ uses $(x,y) = (2,3)$

$$\therefore \quad y^2 = -\frac{1}{2}(2.6)^2 + 11$$

$$y = \sqrt{-\frac{1}{2}(2.6)^2 + 11}$$

$$y = \frac{\sqrt{762}}{10}$$ ✓

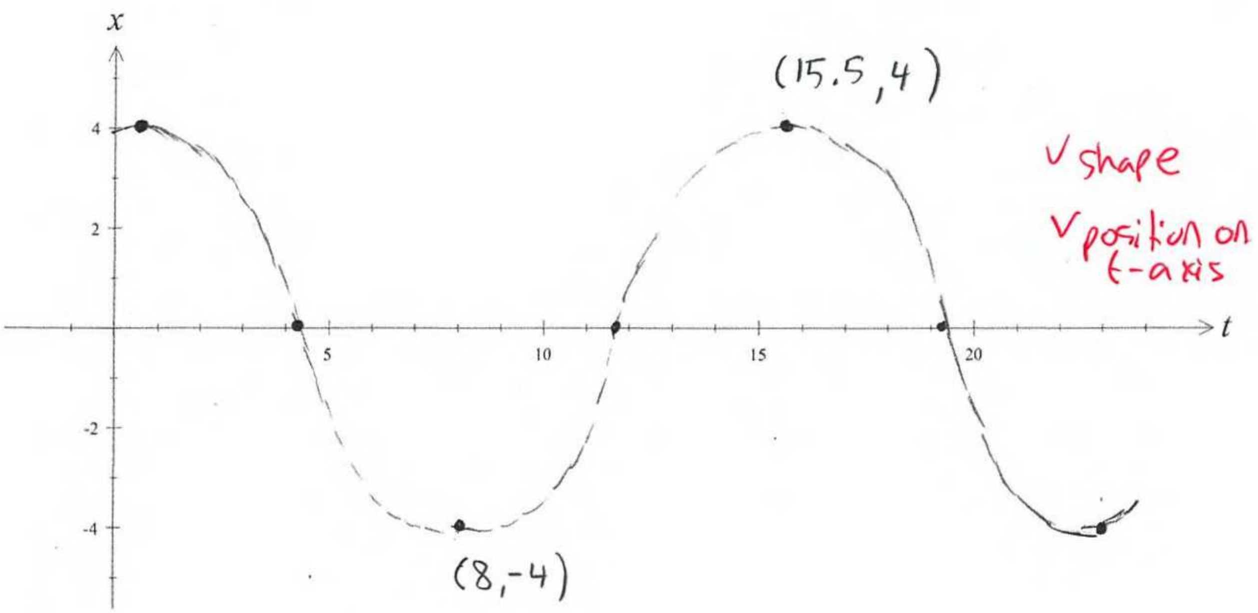
Question 5 **(9 marks)**

The depth of water in a harbour, above and below the mean depth, is an example of simple harmonic motion.

In a particular harbour the low tide depth of 9 metres is recorded at 8 am one morning and the next high tide is expected to record a depth of 17 metres at 3.30 pm later that same day.

A particular container ship requires a depth of at least 11 metres for safe entry into the harbour, for unloading at the dock side and for leaving.

(a) For this particular day, sketch the graph of the depth of water, x (metres), above and below the mean depth, as a function of the time t (hours) in the coordinate system below. (4 marks)



$$\text{Amplitude} = \frac{17-9}{2} = 4 \quad ✓$$

$$\text{period} = 2 \times 7.5 = 15 \quad ✓$$

(b) On this particular day, what proportion of the day will it be safe for the container ship to engage in these activities? (5 marks)

$x(t) = 4\cos\left(\frac{2\pi}{15}(t - 0.5)\right)$, $0 \le t \le 24$ ✓

$x(t) = -2$ ✓ for depth = 11 m

$\Downarrow$

$t = 5.5, 10.5, 20.5$ ✓ intersection

$\therefore$ proportion $= \dfrac{5.5 + (20.5 - 10.5)}{24}$ ✓ determines numerator

$= \dfrac{31}{48}$

≈ 0.65 ✓

Question 6 **(10 marks)**

A TV channel has estimated that if it spends \$$x$ on advertising a particular program it will attract a proportion $y(x)$ of the potential audience for the program, where

$$\frac{dy}{dx} = ay(1-y)$$

for some constant $a > 0$.

(a) Show that $\int \frac{dy}{y(1-y)} = \ln\left(\frac{y}{1-y}\right) + c$ for some constant c, where $0 < y < 1$. (3 marks)

$$\ln\left(\frac{y}{1-y}\right)' = \left(\ln y - \ln(1-y)\right)'$$

$$= \frac{1}{y} - \frac{1}{1-y}(-1)$$ ✓ differentiates correctly

$$= \frac{1-y+y}{y(1-y)}$$ ✓ common denominator

$$= \frac{1}{y(1-y)} \quad \text{as required.}$$ ✓ arrives at result

(b) Hence, or otherwise, show that $y(x) = \frac{1}{ke^{-ax}+1}$ for some constant $k > 0$. (3 marks)

$$\frac{dy}{dx} = ay(1-y)$$

$$\int \frac{1}{y(1-y)}\,dy = \int a\,dx$$ ✓ separate variables

$$\ln\left(\frac{y}{1-y}\right) = ax + c$$

$$\frac{y}{1-y} = ke^{ax}$$ ✓ use $e^c = k$

$$y = (1-y)ke^{ax}$$

$$y(1+ke^{ax}) = ke^{ax}$$

$$y = \frac{ke^{ax}}{1+ke^{ax}} = \frac{1}{Ke^{-ax}+1}$$ ✓ arrives at result

The TV channel knows that if it spends no money on advertising the program then the audience will be one-tenth of the potential audience. It also knows that if it spends $100000 on advertising the program then the audience will be half of the potential audience.

(c) Determine the values of a and k. (4 marks)

$x = 0$: $\dfrac{1}{k+1} = \dfrac{1}{10}$ ✓ use $x=0$

$\therefore k = 9$ ✓

$x = 100000$: $\dfrac{1}{9e^{-100000a}+1} = \dfrac{1}{2}$ ✓ use $x=100000$

$9e^{-100000a} = 1$

$e^{-100000a} = \dfrac{1}{9}$

$-100000a = \ln\left(\dfrac{1}{9}\right)$

$a = \dfrac{\ln 9}{100000}$

$a \approx 0.00002$ ✓

Question 7 **(5 marks)**

A particle is moving in a straight line with its acceleration as a function of x given by $\ddot{x} = -e^{-2x}$, where x is its displacement in metres and $t \geq 0$ is the time in seconds.

It is initially at the origin and is travelling with a velocity of 1 metre per second.

(a) Show that its velocity $\dot{x} = e^{-x}$. (3 marks)

$$\frac{dv}{dt} = -e^{-2x}$$

$$\frac{dv}{dx}\frac{dx}{dt} = -e^{-2x}$$

✓ use chainrule

$$\frac{dv}{dx}v = -e^{-2x}$$

$$\int v\,dv = \int -e^{-2x}\,dx$$

$$\frac{1}{2}v^2 = \frac{1}{2}e^{-2x} + c$$

$\therefore \frac{1}{2} = \frac{1}{2} + c$ (at $t=0, x=0, v=1$), so $c=0$

✓ find c

$\therefore v^2 = e^{-2x}$

$\therefore v = e^{-x}$ since $v > 0$.

✓ arrives at result

(b) Hence show that $x = \ln(t+1)$. (2 marks)

$$\frac{dx}{dt} = e^{-x}$$

$$\int e^x\,dx = \int dt$$

$$e^x = t + c$$

$\therefore 1 = 0 + c$ (at $t=0, x=0$), so $c=1$

✓ find c

$\therefore e^x = t+1$

$\therefore x = \ln(t+1)$

✓ arrives at result

End of questions